BELYI MAPS AND DESSINS D'ENFANTS
LECTURE 9

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CONTENTS

I. REVIEW

Last time we:

(1) Defined covering spaces and the universal covering space of a Riemann surface.
(2) Stated the fact that $\mathrm{Deck}(\widetilde{X}/X) \cong \pi_1(X,x)$, where $p : \widetilde{X} \to X$ is the universal cover.
(3) Defined the monodromy action of $\pi_1(X,x)$ on the fiber of a covering $p : Y \to X$ via path-lifting.

II. COVERING SPACES

Definition 1. Let X be a topological space. A covering space of X is a topological space E together with a continuous map $\pi : E \to X$ called a covering map such that the following property holds. For each $P \in X$ there exists a neighborhood V of P such that $\pi^{-1}(V) = \bigsqcup_i U_i$, where the sets U_i are pairwise disjoint and the restriction $\pi|_{U_i} \to V$ is a homeomorphism. We say that such a neighborhood V is evenly covered by π.

Definition 2. A deck transformation of a covering $\pi : E \to X$ is an automorphism of the covering, i.e., an automorphism $f : E \to E$ such that the following diagram commutes.

$$\begin{array}{ccc} E & \xrightarrow{\ f\ } & E \\ & {\scriptstyle\pi}\searrow \quad \swarrow{\scriptstyle\pi} & \\ & X & \end{array}$$

The set of deck transformations of π is a group, denoted $\mathrm{Deck}(E/X)$ or $\mathrm{Deck}\left(E \xrightarrow{\pi} X\right)$.

Theorem 3. *Let X be a connected Riemann surface. Then there exists a covering $\pi : \widetilde{X} \to X$ with $\widetilde{X}$ connected and simply connected. Moreover $\widetilde{X}$ is unique up to isomorphism.*

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Definition 4. The covering space $\widetilde{X}$ in the previous theorem is called the universal covering space of X.

Theorem 5. *Let X be a connected Riemann surface and $p : \widetilde{X} \to X$ be its universal cover. Then* $\mathrm{Deck}(\widetilde{X}/X) \cong \pi_1(X, x)$ *for any choice of basepoint $x \in X$.*

Lemma 6 (Path-lifting lemma)**.** *Let $p : E \to X$ be a covering space. Let γ be a path on X and let $x = \gamma(0)$. Given any preimage $e \in p^{-1}(x)$ there exists a unique path $\widetilde{\gamma}$ on E such that $p \circ \widetilde{\gamma} = \gamma$ and $\widetilde{\gamma}(0) = e$.*

Definition 7. Such a $\widetilde{\gamma}$ is called a lift of γ based at e.

Lemma 8. *The above definition*

$$\begin{aligned} p^{-1}(x) \times \pi_1(X, x) &\to p^{-1}(x) \\ (\widetilde{x}, [\gamma]) &\mapsto \widetilde{x} \cdot [\gamma] = \widetilde{\gamma}(1) \end{aligned}$$

gives a right $\pi_1(X, x)$ action on $p^{-1}(x)$.

Proof. Note that, due to our convention that $\gamma_1 * \gamma_2$ traverses first γ_1, then γ_2, then

$$\widetilde{x} \cdot [\gamma_1 * \gamma_2] = (\widetilde{x} \cdot [\gamma_1]) \cdot [\gamma_2]$$

so this is a *right* action. □

After converting this into a left action, we get a group homomorphism $\pi_1(X, x) \to \mathrm{Sym}(p^{-1}(x))$. If the fiber $p^{-1}(x)$ is finite, containing d points, then by labeling the points $1, 2, \ldots, d$, we can identify $\mathrm{Sym}(p^{-1}(x)) \cong S_d$, hence we obtain a homomorphism $\pi_1(X, x) \to S_d$.

Definition 9. Let X be a connected Riemann surface, $x \in X$ and let $p : E \to X$ be a covering space. Let $\theta : \pi_1(X, x) \to \mathrm{Sym}(p^{-1}(x))$ be the group homomorphism defined above. Then θ is called the monodromy representation of p and the image of θ is called its monodromy group.

The next result allows us to apply our results on covering spaces to morphisms of Riemann surfaces. Basically, once we throw out the ramification points and values of such a morphism, the resulting map is a covering map, hence all the above results hold for it.

Theorem 10. *Let $\pi : X \to Y$ be a nonconstant morphism of compact Riemann surfaces, and let $\Sigma \subseteq Y$ be its set of ramification values. Let $Y^* := Y \setminus \Sigma$ and $X^* := X \setminus \pi^{-1}(\Sigma)$. Then the restriction $\pi|_{X^*} : X^* \to Y^*$ is an (unramified) covering map.*

III. Groups acting on Riemann surfaces

Definition 11. Let G be a group.

- Let X be a topological space. A (continuous) action of G on X is a group homomorphism $G \to \mathrm{Homeo}(X)$, the group of self-homeomorphisms $X \to X$.
- Let X be a Riemann surface and G be a group. A (holomorphic) action of G on X is a group homomorphism $G \to \mathrm{Aut}(X)$.

Given a group G acting on a Riemann surface X, we can form the quotient space $G\backslash X$ whose points are the G-orbits of X. There is a natural quotient map

$$\begin{aligned} \pi : X &\to G\backslash X \\ x &\mapsto [x] \end{aligned}$$

where $[x]$ denotes the G-orbit of x. Without further restrictions, $G\backslash X$ will only be a topological space, not necessarily a Riemann surface. The following properties of group actions yield nice properties of the quotient space $G\backslash X$ and the quotient map π.

Definition 12. Let G be a group acting (holomorphically) on a Riemann surface X.

(a) The action is faithful (or effective) if the kernel of the homomorphism $G \to \mathrm{Aut}(X)$ is trivial.

(b) The action is free if for all points $x \in X$, the stabilizer

$$\mathrm{Stab}_G(x) := \{g \in G : g \cdot x = x\}$$

is trivial.

(c) The action is properly discontinuous or wandering if, for each $x \in X$ there exists an open neighborhood $U \ni x$ such that the set

$$\{g \in G : gU \cap U \neq \varnothing\}$$

is finite. In particular, this means that $\mathrm{Stab}_G(x)$ is finite for all $x \in X$.

Lemma 13. *If G acts on X properly discontinuously, then $G\backslash X$ is Hausdorff.*

Proposition 14. *Suppose G is a group acting on X freely and properly discontinuously. Then the quotient map $\pi : X \to G\backslash X$ is a covering map with deck transformation group G.*

We now have the language to define the Galois correspondence given by the universal covering space.

Theorem 15. *Let X be a connected Riemann surface and $p : \widetilde{X} \to X$ be its universal cover.*

(a) The action of $\mathrm{Deck}(\widetilde{X}/X)$ on $\widetilde{X}$ is free and properly discontinuous. Moreover, the action is transitive on each fiber.

(b) The action induces an isomorphism of Riemann surfaces

$$\begin{aligned} \mathrm{Deck}(\widetilde{X}/X)\backslash\widetilde{X} &\to X \\ [\tilde{x}] &\mapsto p(\tilde{x})\,. \end{aligned}$$

(c) Let $q : E \to X$ be a covering. Then there exists a subgroup $H \leq \mathrm{Deck}(\widetilde{X}/X)$ such that $E \cong H\backslash\widetilde{X}$ as Riemann surfaces, and the following diagram commutes

$$\begin{array}{ccc} E & \xrightarrow{\sim} & H\backslash\widetilde{X} \\ {\scriptstyle q}\downarrow & & \downarrow \\ X & \xrightarrow{\sim} & \mathrm{Deck}(\widetilde{X}/X)\backslash\widetilde{X} \end{array}$$

Remark 16. Parts (b) and (c) of the above theorem should remind you of Galois theory.

$$\begin{array}{ccc} K & & 1 \\ \Big| & & \Big| \\ E = K^H & & H \\ \Big| & & \Big| \\ F & & \mathrm{Gal}(K/F) \end{array}$$

$$\begin{array}{ccc} \widetilde{X} & & 1 \\ \downarrow & & \Big| \\ E = H\backslash\widetilde{X} & & H \\ \downarrow & & \Big| \\ X & & \mathrm{Deck}(\widetilde{X}/X) \end{array}$$

Proposition 17. *Let X be a Riemann surface and $x \in X$. Then there is a bijective correspondence*

$$\left\{ \begin{array}{c} \text{isomorphism classes of} \\ \text{connected coverings} \\ F : E \to X \end{array} \right\} \overset{\sim}{\longleftrightarrow} \left\{ \begin{array}{c} \text{conjugacy classes of} \\ \text{subgroups } H \leq \pi_1(X, x) \end{array} \right\}.$$

Example 18. Consider the universal cover $p : \mathbb{R} \to \mathbb{S}^1$, $p : t \mapsto e^{2\pi i t}$ and the covering $q : \mathbb{S}^1 \to \mathbb{S}^1$, $q : z \mapsto z^2$. Let $H \leq \mathrm{Deck}(\mathbb{R}/\mathbb{S}^1)$ be the subgroup of all deck transformations of the form $\alpha : s \mapsto s + 2m$ for some $m \in \mathbb{Z}$. Then

$$2\mathbb{Z} \cong H \leq \mathrm{Deck}(\mathbb{R}/\mathbb{S}^1) \cong \mathbb{Z}$$

so $[\mathrm{Deck}(\mathbb{R}/\mathbb{S}^1) : H] = 2$, and $H\backslash\mathbb{R} \cong 2\mathbb{Z}\backslash\mathbb{R} \cong \mathbb{S}^1$. (We're identifying every other loop in the helix, so we get 2 coils of the helix after quotienting.) Moreover, we get a commutative diagram.

$$\begin{array}{ccccc} z & & \mathbb{S}^1 & \xrightarrow{\sim} & H\backslash\mathbb{R} = 2\mathbb{Z}\backslash\mathbb{R} \\ \Big\downarrow & & {\scriptstyle q}\Big\downarrow & & \Big\downarrow \\ z^2 & & \mathbb{S}^1 & \xrightarrow{\sim} & \mathbb{Z}\backslash\mathbb{R} \end{array}$$

Theorem 19. *Let G be a finite group acting faithfully on a Riemann surface X. Then $G\backslash X$ can be given the structure of a Riemann surface, and the quotient map $\pi : X \to G\backslash X$ is holomorphic of degree $\#G$ and $e_P(\pi) = \#\mathrm{Stab}_G(P)$ for all $P \in X$.*

Corollary 20. *Let G be a finite group acting faithfully on a compact, connected Rieman surface X, let $Y = G\backslash X$ and let $\pi : X \to Y$ be the quotient map. Suppose that π has k ramification values $y_1, \ldots, y_k \in Y$ such that π has ramification index r_i at each of the $\#G/r_i$ points above y_i.*

Then

$$\begin{aligned} 2g(X) - 2 &= \#G(2g(G\backslash X) - 2) + \sum_{i=1}^{k} \frac{\#G}{r_i}(r_i - 1) \\ &= \#G\left(2g(G\backslash X) - 2 + \sum_{i=1}^{k}\left(1 - \frac{1}{r_i}\right)\right). \end{aligned}$$

Proof. [This follows from the above theorem, and which previous result?] ☐

Example 21. Let $E : y^2 = f(x)$ be an elliptic curve, and $\iota : (x,y) \mapsto (x,-y)$ be the hyperelliptic involution. Then $\langle\iota\rangle \leq \mathrm{Aut}(E)$ is a subgroup of order 2, so $G = \mathbb{Z}/2\mathbb{Z}$ acts on E via ι. Let $\alpha_1, \alpha_2, \alpha_3$ be the roots of f. Then the only points of E fixed by ι are $(\alpha_1, 0), (\alpha_2, 0), (\alpha_3, 0)$ and ∞. By the above theorem and corollary, then

$$0 = 2g(E) - 2 = 2\left(2g(G\backslash X) - 2 + \sum_{i=1}^{4}(1 - 1/2)\right) = 4g(G\backslash X) \implies g(G\backslash X) = 0$$

so $G\backslash X \cong \mathbb{P}^1$.

This matches our intution as well: quotienting out by $\langle\iota\rangle$ identifies points with the same x-coordinate. Thus points in $G\backslash X$ are determined by their x-coordinate, so the quotient map $\pi : X \to G\backslash X$ is really just the projection $(x,y) \mapsto x$.

Lemma 22. *Suppose that $r_1, \ldots, r_k \in \mathbb{Z}_{\geq 2}$, and let $R = \sum_{i=1}^{k}\left(1 - \frac{1}{r_i}\right)$. Then*

(a) $R < 2 \iff \begin{cases} k = 1, & \text{any } r_1\text{; or} \\ k = 2, & \text{any } r_1, r_2\text{; or} \\ k = 3, & r_1 = 2, r_2 = 2, \text{any } r_3\text{; or} \\ k = 3, & \{r_1, r_2, r_3\} = \{2,3,3\}, \{2,3,4\} \text{ or } \{2,3,5\}. \end{cases}$

(b) $R = 2 \iff \begin{cases} k = 3, & \{r_1, r_2, r_3\} = \{2,3,6\}, \{2,4,4\}, \text{ or } \{3,3,3\}\text{; or} \\ k = 4 & \{r_1, r_2, r_3, r_4\} = \{2,2,2,2\}. \end{cases}$

(c) If $R > 2$, then in fact $R \geq 2 + \frac{1}{42}$.

Remark 23. Case (b) is reminiscent of what geometric phenomenon?

Theorem 24. *Let G be a finite group acting faithfullly on a compact, connected Riemann surface X of genus $g \geq 2$. Then*

$$\#G \leq 84(g-1).$$

Proof. By the above corollary, we have

$$2g - 2 = \#G(2g(G\backslash X) - 2 + R)$$

where $R = \sum_{i=1}^{k}\left(1 - \frac{1}{r_i}\right)$. Note that the lefthand side is ≥ 2 since $g \geq 2$.

Case 1: Suppose that $g(G\backslash X) \geq 1$.

- If $R = 0$, since $2g - 2 > 0$, then $g(G\backslash X) \geq 2$. Then
$$2g - 2 = \#G(2g(G\backslash X) - 2) \geq \#G \cdot 2$$
so $\#G \leq g - 1$.
- If $R \neq 0$, then $r_i \geq 2$ for some i, so $R \geq 1/2$. Then $2g(G\backslash X) - 2 + R \geq 1/2$, so
$$2g - 2 = \#G(2g(G\backslash X) - 2 + R) \geq \#G \cdot (1/2)$$
so $\#G \leq 4(g - 1)$.

<u>Case 2:</u> Now suppose that $g(G\backslash X) = 0$. Then
$$2g - 2 = \#G(-2 + R)$$
so we must have $R > 2$. By part (c) of the above lemma, then $R \geq 2 + \dfrac{1}{42}$, so
$$2g - 2 = \#G(-2 + R) \geq \#G \cdot \frac{1}{42}$$
and thus $\#G \leq 84(g - 1)$. □

Remark 25. In fact, $\operatorname{Aut}(X)$ is a finite group for all Riemann surfaces of genus $g \geq 2$. (We may prove this later.) This fact, combined with the above theorem, is known as *Hurwitz's automorphisms theorem*, which states that $\# \operatorname{Aut}(X) \leq 84(g - 1)$.

IV. More monodromy

Lemma 26. *Let X, Y be topological spaces, let $p : X \to Y$ be a covering map of finite degree, and let $\rho : \pi_1(Y, y) \to S_d$ be its associated monodromy representation. If X is path-connected, then the image of ρ is a transitive subgroup of S_d.*

Proof. Recall that a transitive subgroup of S_d is one that acts transitively on $\{1, 2, \ldots, d\}$. Fix indices i and j, and let x_i, x_j be the corresponding points in the fiber $p^{-1}(y)$. Since X is path-connected, then there exists a path δ starting x_i and ending at x_j. Letting $\gamma = F \circ \delta$, then γ is a loop in Y based at y. Moreover, by uniqueness the lift of γ starting at x_i must be δ, so $\rho([\gamma])$ maps i to j. □

Example 27. Let $\mathfrak{D}^* := \mathfrak{D} \setminus 0$ be the punctured unit disc, considered as a subset of $\mathbb{C}$. Let $p : \mathfrak{D}^* \to \mathfrak{D}^*$ be the covering map given by $w \mapsto w^d$ for some $d \in \mathbb{Z}_{\geq 1}$. Take $z_0 = 1/2^d$ as the basepoint of the codomain. Letting ζ be an primitive d^{th} root of unity, then $p^{-1}(z_0)$ consists of the points $x_j := \zeta^j/2$ for $j = 1, \ldots, d$.

Letting $\gamma : [0, 1] \to \mathfrak{D}^*$, $\gamma(t) = \dfrac{1}{2^d} e^{2\pi i t}$, then $[\gamma]$ is a generator for $\pi_1(\mathfrak{D}^*, z_0)$. The loop γ lifts to the loops $\widetilde{\gamma}_j : [0, 1] \to \mathfrak{D}^*$ given by $\widetilde{\gamma}_j(t) = \zeta^j \dfrac{1}{2} e^{2\pi i t/d}$, whose starting point is $\zeta^j/2$ and whose ending point is $\zeta^{j+1}/2$. Thus the monodromy representation $\rho : \pi_1(\mathfrak{D}^*, z_0) \to S_d$ sends $[\gamma]$ to the cyclic permutation that takes j to $j + 1$, i.e.,
$$\rho([\gamma]) = (1\ 2\ \cdots\ d)\,.$$

By the Local Normal Form theorem, every morphism of Riemann surfaces locally looks like $z \mapsto z^d$, so our above example is actually quite general.

We now discuss the monodromy of a morphism $F : X \to Y$ of degree d of compact, connected Riemann surfaces. Let $\Sigma \subseteq Y$ be its set of ramification values and let $Y^* := Y \setminus \Sigma$ and $X^* := X \setminus \pi^{-1}(\Sigma)$. As we saw previously, then the restriction $F|_{X^*} : X^* \to Y^*$ is an (unramified) covering map. The monodromy representation of F is defined to be the monodromy representation $\rho : \pi_1(Y^*, y) \to S_d$ of this resriction. Since X is connected, then $\operatorname{img}(\rho) \leq S_d$ is a transitive subgroup.

Lemma 28. *With notation as above, suppose above a ramification value $b \in Y$ there are k preimages $u_1, \ldots, u_k \in F^{-1}(b)$, with ramification indices $e_i := e_{u_i}(F)$. Then the permutation σ representing a small loop around b has cycle structure $(m_1, \ldots, m_k)$, i.e., it is composed of k disjoint cycles of lengths $m_1, \ldots, m_k$.*

Proof. Let $y \in Y$ be a basepoint. Fix a ramification value $b \in Y$ and choose a small open neighborhood W of b that is isomorphic to the open disc $\mathfrak{D}$. Let $u_1, \ldots, u_k$ be the points in the fiber $F^{-1}(b)$; since b is a ramification value, then at least one of the u_j must be a ramification point.

Choose W small enough such that $F^{-1}(W \setminus \{b\})$ decomposes as a disjoint union of open punctured neighborhoods $U_1, \ldots, U_k$ of $u_1, \ldots, u_k$, respectively. Letting $e_j = e_{u_j}(F)$, then by the Local Normal Form Theorem, there are coordinates z_j on U_j and z on W such that F locally has the form $z_j \mapsto z_j^{m_j}$.

Then F sends $U_j \setminus \{u_j\}$ to $W \setminus \{b\}$ via the m_j^{th} power map. Choose a path α from the basepoint y to a point $y_0 \in W \setminus \{b\}$, and let β be a loop in $W \setminus \{b\}$ based at y_0 that winds once around the ramification value b. Then the path $\gamma := \alpha^{-1} * \beta * \alpha$ is a loop in Y based at y, which we will call a small loop on Y around b. Since F is an unramified covering away from Σ, then the path α simply gives a bijection between the fibers $F^{-1}(y)$ and $F^{-1}(y_0)$. Thus the permutation σ of the fiber $F^{-1}(y)$ is determined up to this identification by the loop β around b.

Above the punctured neighborhood $W \setminus \{b\}$ we have k punctured discs $U_j \setminus \{u_j\}$, each mapping to $W \setminus \{b\}$ via the m_j^{th} power map. By the example above, the monodromy for each covering $F|_{U_j \setminus \{u_j\}} : U_j \setminus \{u_j\} \to W \setminus \{b\}$ is a cyclic permutation of the m_j preimages of y_0 which lie in U_j. Thus the loop β based at y_0 and hence the loop γ based at y induce cyclic permutations of the points above y, and the cycle corresponding to u_j has length m_j. $\square$